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LETTER TO THE EDITOR

Two-dimensional turbulence with the lattice Boltzmann equation

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Abstract. We investigate the ability of the lattice Boltzmann equation to reproduce the basic physics of fully turbulent two-dimensional flows and present a qualitative estimate of its computational efficiency with respect to other conventional techniques.

There is growing theoretical and numerical evidence that lattice gas automata (LGA) simulate the dynamics of turbulent flows quite accurately in the sense that they give the same quantitative results as compared with direct numerical simulations of the Navier-Stokes equations. LGA can enhance our understanding of physics in a variety of different problems, ranging from non-equilibrium statistical mechanics to the dynamics of fluid flows in very complex geometries. On the other hand, regarding turbulent flows in a relatively simple geometry, LGA are not computationally as efficient, at least for general purpose computers, as other well established numerical techniques like finite differences or spectral methods.

Recently, McNamara and Zanetti (1988) and Higuera and Jimenez (1989) have pointed out that the lattice Boltzmann equation (LBE) can be a possible efficient tool to simulate two- and three-dimensional flows. Pursuing this idea, Higuera *et al* (1989) have improved this approach, showing that it is possible to generalise the LBE for arbitrary values of the Navier-Stokes parameters in order to increase the efficiency of the method. In this letter we investigate this idea on quantitative grounds, in order to estimate the efficiency of the LBE and also to understand its intrinsic limitations. To this end, we have chosen the case of two-dimensional flows contained in a square box with periodic boundary conditions. The efficiency of the LBE has been tested against spectral methods, which are known to be probably the most efficient method for this kind of problem.

According to Higuera *et al* (1989), we can write the LBE in the following way:

$$N_i(t+1, x+c_i) = N_i(t, x) + \sum_j A_{i,j}\left(N_j(x) - \frac{\rho}{b} - G(\rho) \sum_{\alpha,\beta} Q_{j,\alpha\beta} u_\alpha u_\beta \right) + F_i \tag{1}$$

where

$$G(\rho) = \frac{\rho D^2 (b-2\rho)}{2c^4 b(b-\rho)} \qquad Q_{i,\alpha\beta} = c_{i\alpha} c_{i\beta} - \frac{c^2}{D}\delta_{\alpha\beta}.$$

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In equation (1) N_i is the mean particle population in the ith state; $\rho = \sum_i^b N_i$ is the density of the fluid; $\rho u_\alpha = \sum_i^b c_{i\alpha} N_i$ is the momentum density; b is the number of automata states per site; D is the dimension of the lattice and F_i is an external force acting on the system (for a review of LGA see Frisch *et al* 1986, 1987). We have considered the case of a four-dimensional face-centred hypercubic lattice (FCHC) projected into two dimensions, in which case $b = 24$. Equation (1) is obtained by expanding the nonlinear collision operator of the LBE up to second order in the velocity field (for details see Higuera and Jimenez (1989) and McNamara and Zanetti (1988)). The matrix $A_{i,j}$ satisfies the constraint of mass conservation ($\sum_j A_{i,j} = 0$) and momentum conservation ($\sum_j A_{i,j} c_j = 0$). Because of these constraints, and the symmetries of the lattice, $A_{i,j}$ has only three independent eigenvalues, λ, τ and σ, which can be chosen arbitrarily. In particular, the viscosity of the fluid depends on λ only through the relation $\nu = (2+\lambda)/6\lambda$. After projection of the original four-dimensional lattice into two dimensions, only nine independent populations are involved in the computations.

Following Frisch *et al* (1987), from equation (1) one can derive the equations

$$\frac{1}{g}\frac{\partial}{\partial t_a} \boldsymbol{V}_a + \boldsymbol{V}_a \cdot \boldsymbol{\partial} \boldsymbol{V}_a = -\frac{1}{\rho_a} \boldsymbol{\partial} p_a + \frac{\nu_a}{g} \Delta_a V_a + \frac{\boldsymbol{F}_a}{g} \tag{2}$$

where the index a refers to the automata units and $g = G/3$. By performing the rescaling:

$$V_a = U_0 V \tag{3a}$$

$$x_a = l_0 x \tag{3b}$$

$$t_a = \frac{l_0}{g U_0} t \tag{3c}$$

we finally obtain the Navier-Stokes equations:

$$\partial_t \boldsymbol{V} + \boldsymbol{V} \cdot \boldsymbol{\partial} \boldsymbol{V} = -\frac{1}{\rho} \boldsymbol{\partial} p + \nu \Delta \boldsymbol{V} + \boldsymbol{F}. \tag{4}$$

As a first insight into the efficiency of the method, we have compared the numerical results of (1) against a spectral simulation of (4). For both cases we have chosen $\nu = 0.05$ corresponding to $\lambda = -1.849$. The resolution is 64×64 and the forcing term is $\boldsymbol{F} = (\cos 4x, 0)$. This value implies $l_0 = \frac{64}{2}\pi = 10.18$ and $U_0 = \frac{1}{25}$. The time step for the spectral simulation is 0.01 and one step in (1) corresponds to $\frac{1}{382}$ time units of (4) according to (3c). The time marching scheme for the spectral code is a predictor-corrector Eulero-Cauchy method. No de-aliasing has been performed in the spectral simulation.

The time series of the energy E and the enstrophy Ω for both the LBE and the Navier-Stokes equation are shown in figure 1. As we can see, both average and fluctuations of E and Ω agree quite well. In order to get a precise idea of the accuracy of the method we have performed a numerical simulation at a resolution 128×128 with the same values for the other parameters. In figure 2 we show the energy spectrum for both the LBE and the spectral simulation. Because the resolution is much larger than what is needed for this value of the Reynolds number, at very high wavenumbers we observe a rather flat spectrum. The important point is that the beginning of this flat spectrum is at the same wavenumber for both cases.

Having tested the numerical accuracy of the algorithm, we now discuss some computational details. The number of floating point operations for one step of the

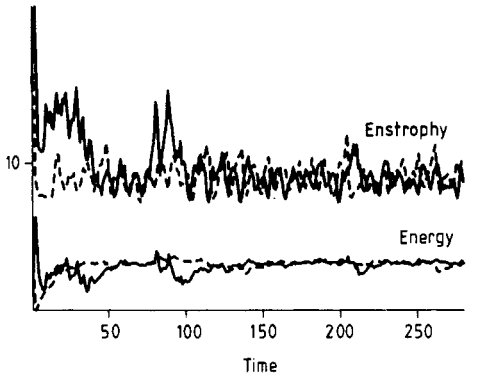

Figure 1. Time evolution of the total energy and enstrophy for the spectral (broken curve) and LBE (full curve) models.

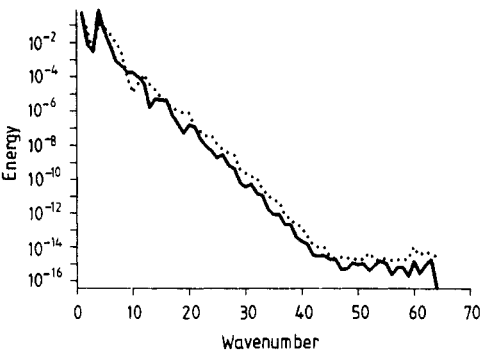

Figure 2. Energy spectra of the spectral (full curve) and LBE (broken curve) simulations on a 128^2 grid.

LBE is about $150N^2$ where N is the linear size of the domain. For the spectral simulation one time step, without de-aliasing, requires approximately $50N^2 \log_2 N$ operations. The time step for the LBE in physical units is $\Delta t_a = \frac{1}{382}$ while for the spectral method we have $\Delta t_s = 0.01$. On a single processor of the IBM 3090 vector multiprocessor we measured about 60 ms/step for the LBE and about 300 ms/step for the spectral method (no de-aliasing and two time step-marching scheme) at a resolution of 128^2. Taking into account the differences between the time steps for the two cases, we find that almost the same computational work is required for both algorithms. However, for increasing values of N, the LBE should eventually become more efficient than the spectral method because of its favourable scaling (N^2 instead of $N^2 \log_2 N$).

Another important point to be discussed is the performance of LBE in dealing with very high numerical resolutions, i.e. at high Reynolds number. In this case the main question to be answered is whether or not the LBE is able to reproduce the statistical properties of two-dimensional turbulence, namely the enstrophy inertial range at high wavenumber. Actually the detailed knowledge of the enstrophy inertial range is still a matter of discussion in the recent literature (Benzi *et al* 1986, Legras *et al* 1988, Brachet *et al* 1986).

We have performed a high numerical resolution (at 512×512) to study whether or not an enstrophy inertial range can be detected by using the LBE. The forcing is the same as that used in the previous experiments, while the Reynolds number is now increased by a factor of 7. In figure 3 we show the energy spectrum of our numerical results and in figure 4 we show the corresponding vorticity map. As one can see, no clear evidence of an inertial range can be observed in the spectrum, although a slope near k^{-4}, k^{-5} could eventually be measured. This slope is much steeper than what has been observed in previous experiments at the same numerical resolution (Legras *et al* 1988). A possible explanation of this disagreement is the following. Legras *et al* (1988) used a superviscosity proportional to Δ^8 of the velocity field and an energy dissipation mechanism of the large scales, namely a term proportional to Δ^{-1} of the vorticity. The superviscosity increases the effective resolution of the numerical experiments by pushing the dissipation range towards very large values of k. On the other hand, energy dissipation prevents energy pile-up towards low wavenumbers. Neither mechanism is included in our case. In particular, we think that the superviscosity is a rather crucial mechanism in order to obtain a well defined inertial range and we do not see how to

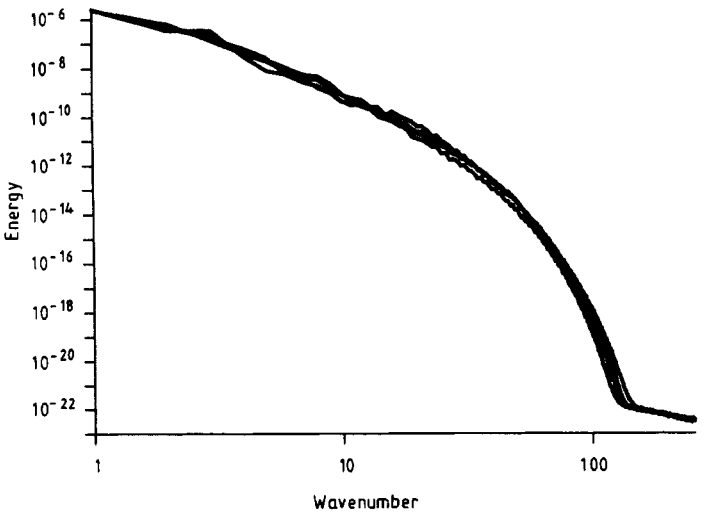

Figure 3. Energy spectra of the LBE simulation on a 512^2 grid.

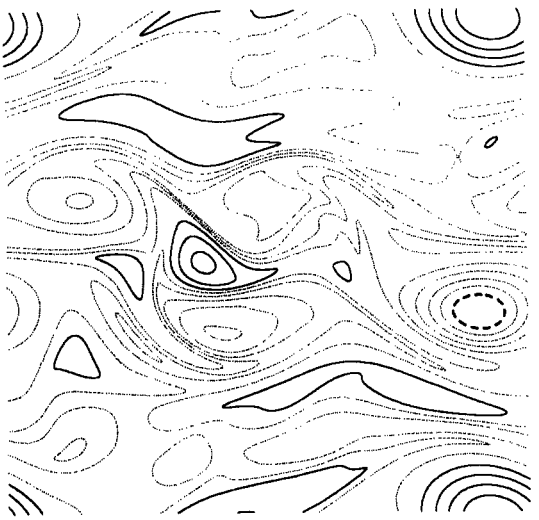

Figure 4. Instantaneous vorticity map for a 512^2 LBE simulation.

include such an effect in the LBE in a simple way. In our opinion, this is a rather clear limitation of the LBE, even though further theoretical work could improve the present situation. The piling up of energy at low wavenumbers can clearly be observed both from figures 3 and 4. In particular, in figure 4 we see a few major vortices and a number of vorticity lines showing well defined cusps: the large-scale vortices are consequences of the inverse energy cascade at low wavenumbers; cusps are regions of the fluid where enstrophy cascade to large wavenumbers is active. Therefore, despite the apparent disagreement between our results and those obtained by Legras *et al* (1988), the overall picture of our numerical simulations is consistent with the known phenomenology of two-dimensional turbulence.

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